Motivation

Why do we study median graphs?

- They first appeared in the study of distributive lattices

X set $\quad (\mathcal{P}(X), \cap, \cup) \leftarrow$ distributive lattice
$\subseteq$; $\cap$ gcd ; $\cup$ lcm

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
&
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

- phylogenetics (study of relations between living/living or living/extinct species.
- social choice theory & voting systems.
- Median graphs the 1-skeleton of CAT(0) cube complexes.
  - median graphs ~~easier~~ to deal with than CAT(0) cube complexes
  Sageev: CAT(0) geometry is given by the behaviour and interaction of hyperplanes in these complexes.
  - isometries of median graphs are more understood than those of CAT(0) cube complexes, for the reason that:

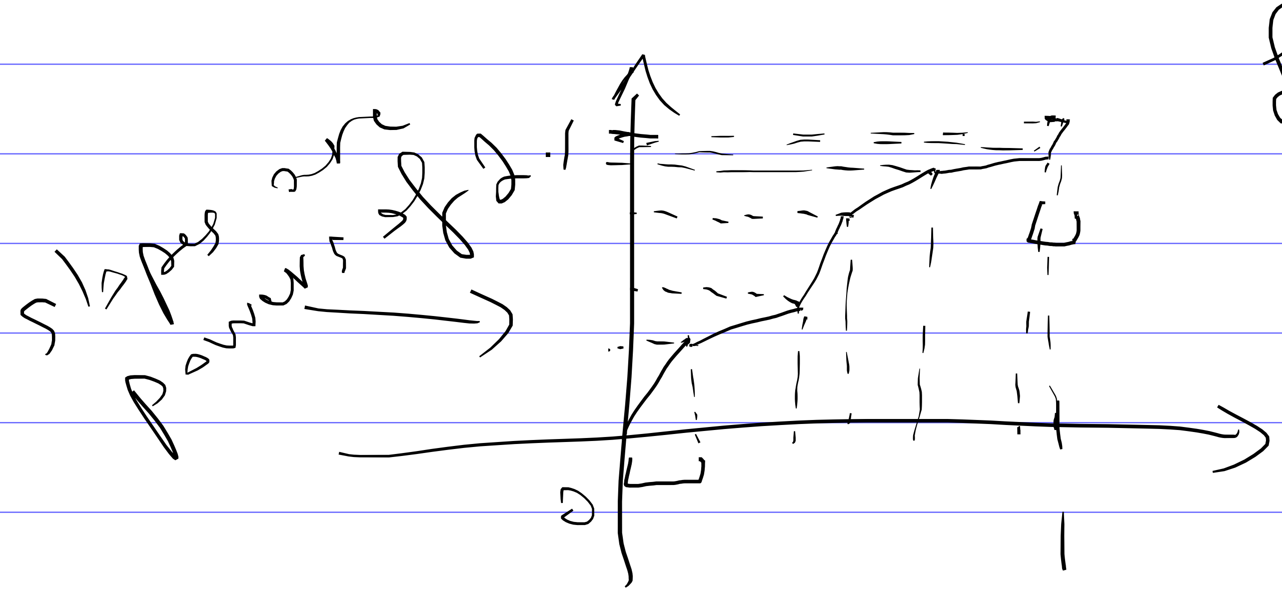
Isometries	Median graphs	CCCAT(0)
	→ periodic (elliptic)	→ periodic
	→ loxodromic (hyperbolic)	→ loxodromic
		→ parabolic

Bridson

g periodic: $g \cdot x = x \quad (\exists x \in X)$

g loxodromic: $\exists\, \gamma$ axis in the complex (or graph) on which it acts as a translation

g parabolic: the rest (classification of the isometries of the Thompson group $F \curvearrowright$ CCCAT(0))



$f : [0,1] \to [0,1]$

$f(0) = 0$

$f(1) = 1$

piecewise affine

increasing

At most countably infinitely many breaking points which happen at dyadic numbers between 0 and 1.

$\frac{m}{2^n}$ $m, n \in \mathbb{N}$.

$g \in F$ is an isometry

g is parabolic iff $\underbrace{g'(0^+) \times g'(1^-) \neq 1}$.

$F \curvearrowright$ median graph $\leftarrow$ $\times$ parabolic isometry

Refer to Dan Titus Salajan : CAT(0) geometry for the Thompson group.

→ Algebraic consequences on the group acting on a median graph

→ G free $\Leftrightarrow$ $G \overset{\text{free}}{\curvearrowright}$ tree $\subset$ median graph

→ $G \overset{\text{proper}}{\curvearrowright} X$ median graph

proper action.

g loxodromic

$g^p = t g^q t^{-1} \Leftrightarrow p = \pm q$

$\Downarrow$

$G \not\geq BS(n,m) = \langle a, t \mid t a^n t^{-1} = a^m \rangle$

if $m \neq -n$

→ **a-t-menability** : (Haagerup property)

$G \overset{\text{proper}}{\curvearrowright} X$ median graph

$\Downarrow$

G has the Haagerup property

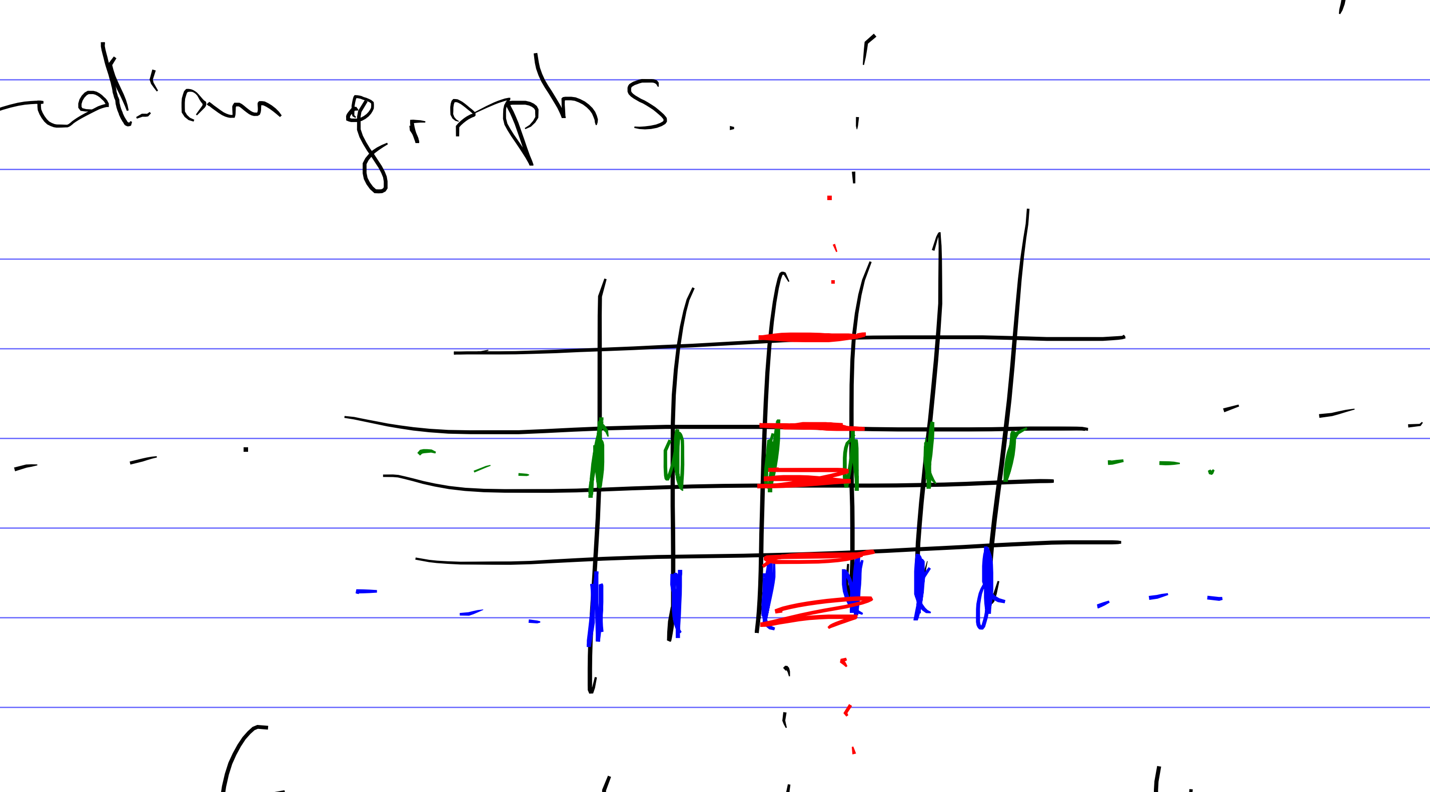
$\Longleftrightarrow$ $G \curvearrowright \mathcal{H}$ Hilbert space
by proper continuous affine isometries.

e.g.: the case of the Thompson group F

We can construct a median graph on which F acts properly $\Rightarrow$ F is a-t-menable.

Last time we talked about hyperplanes in median graphs.

$\mathbb{E}^2$



Theorem: (The geometry of median graphs reduces to the combinatorics of its hyperplanes)

X median graph, then:

(1) Every hyperplane delimits exactly two halfspaces.

(2) Carriers, fibres and halfspaces are all convex subgraphs of X.

(3) Every geodesic in X cross each hyperplane at most once.

(4) The distance between two points in X is equal to the number of hyperplanes separating them.

Positions of two hyperplanes:

Notation

$\mathcal{H}(X) := \{$ hyperplanes in $X\}$

X median, $J_1, J_2 \in \mathcal{H}(X)$

(1) Transversality

J_1 and J_2 are transverse ($J_1 \perp J_2$)

iff J_1 and J_2 intersect and span a 4-cycle.

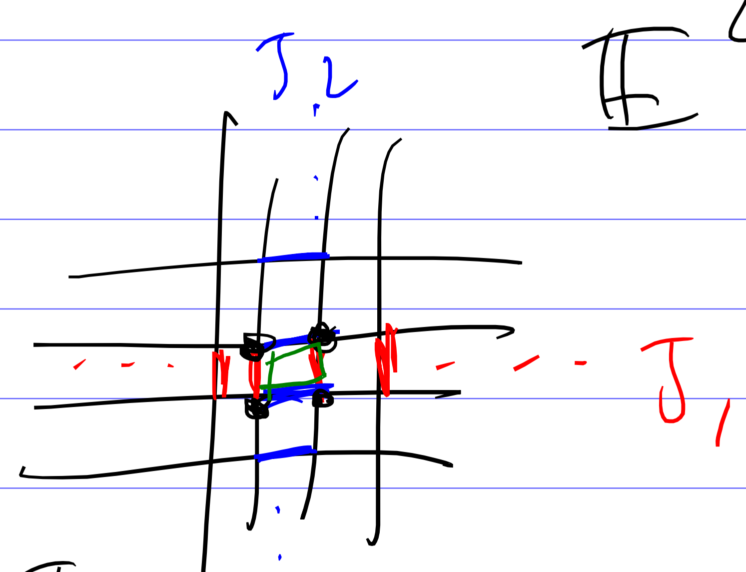
(2) Parallelism:

J_1 and J_2 are parallel iff ($J_1 \parallel J_2$)

iff J_1 and J_2 intersect but do not span a 4-cycle.

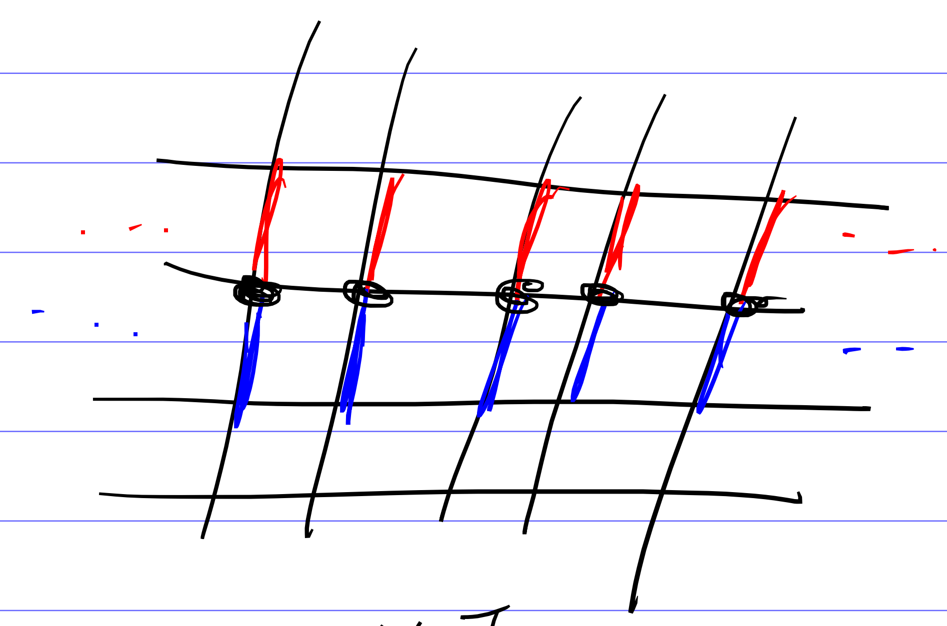
e.g.:

$\mathbb{H}^2$



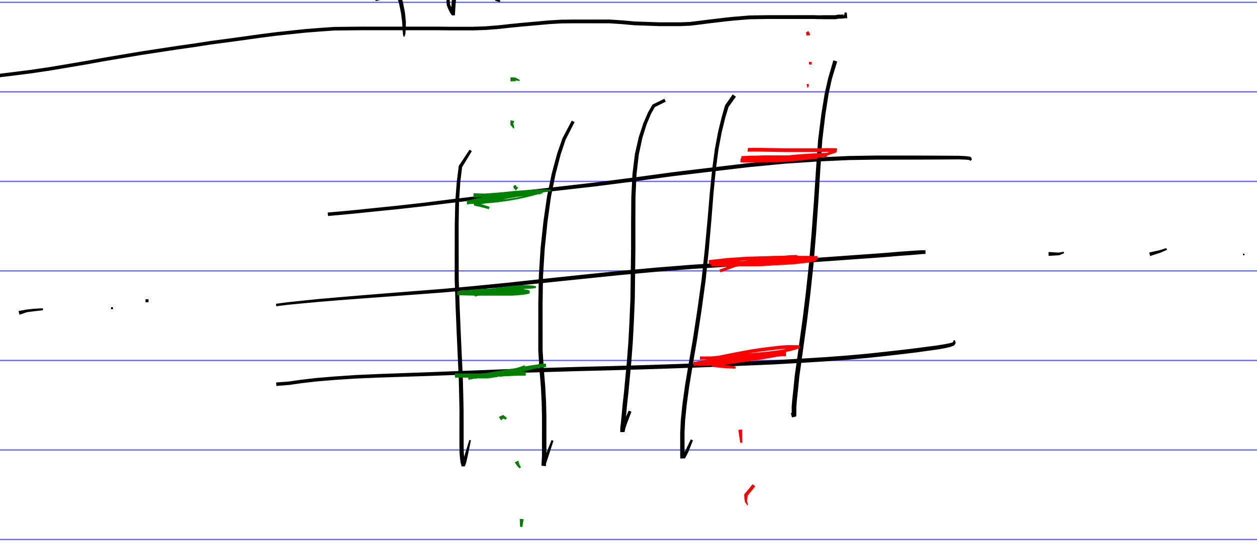
$J_1 \perp J_2$

$\mathbb{E}^2$



$J_1 \parallel J_2$

Neither parallel nor transverse



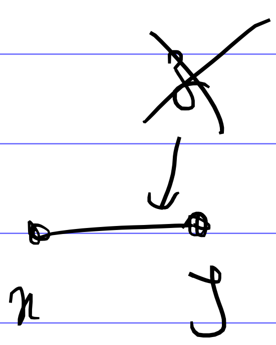
Interesting properties about hyperplanes:

(1) A hyperplane cannot be transverse to itself.

(2) J_1 and J_2 transverse iff $\begin{cases} J_1^+ \cap J_2^+ \neq \emptyset \\ J_1^- \cap J_2^- \neq \emptyset \\ J_1^+ \cap J_2^- \neq \emptyset \\ J_1^- \cap J_2^+ \neq \emptyset \end{cases}$

(3) A subgraph D of X is a halfspace for some hyperplane $J \in \mathcal{H}(X)$ iff D and D^c are both convex.

Projections on convex subgraphs



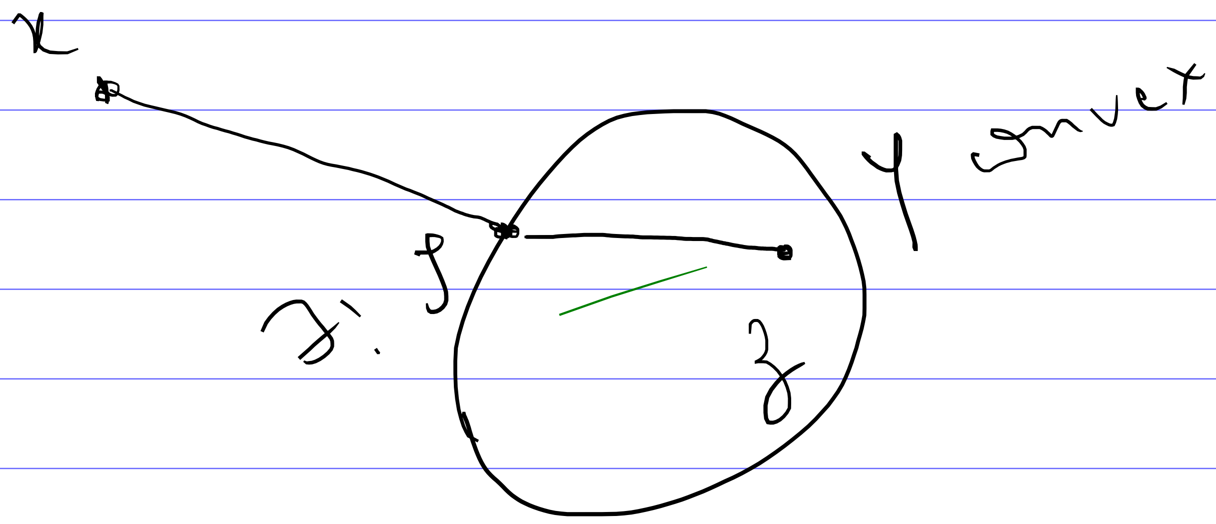
Theorem: Let X be a median graph.
Let $x \in X$ and $\underset{\text{subgraph}}{Y \subseteq X}$ convex.

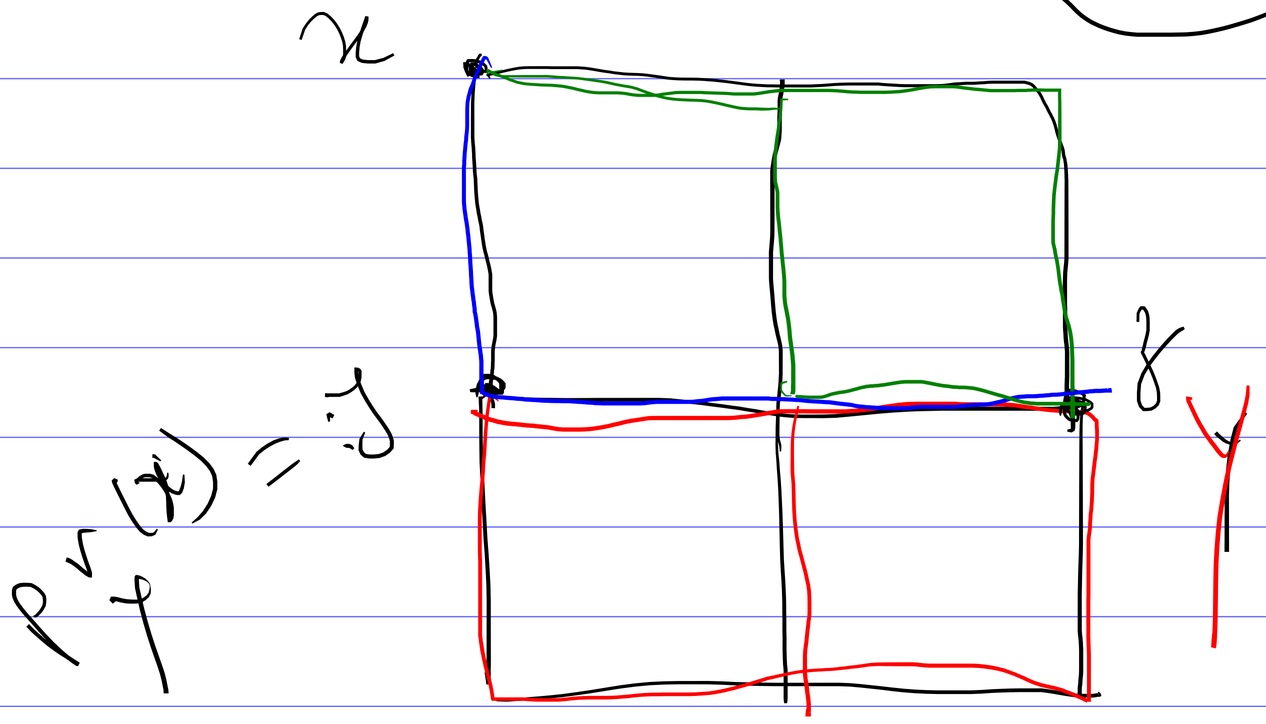
Then there exists a unique vertex $y \in Y$ minimizing the distance from x to Y.

Moreover, for every $z \in Y$, there exists a geodesic from x to z passing through y.

X median

$[x,y] \cup [y,z]$ = geodesic.





<u>Pf. of the theorem above</u>

$x \in X$

Y convex.

Let $y_1, y_2 \in Y$ minimizing the distance to x.

Let $m := \mu(x, y_1, y_2)$.

$$m \in [y_1, y_2] \overset{+Y \text{ is convex}}{\Longrightarrow} [y_1, y_2] \subseteq Y$$
$$\Rightarrow m \in Y.$$

$$m \in [x, y_1] \Rightarrow d(x, y_1) = d(x, m) + d(m, y_1) \quad (*)$$
$$\Rightarrow d(x, m) \leq d(x, y_1)$$

But $\Rightarrow d(x, y_1) \leq d(x, m)$ $\Big\}$ + (*)

$$\Rightarrow d(m, y_1) = 0 \Rightarrow m \equiv y_1$$

Again $\Rightarrow m \equiv y_2$

$\Rightarrow y_1 \equiv y_2 \Rightarrow \exists!\, y \in Y$ minimising $d(x, Y)$.

Let $z \in Y$.

$m := \mu(x, y, z)$.

$m \in [y, z] \subseteq Y$ + Y convex $\Rightarrow m \in Y$.

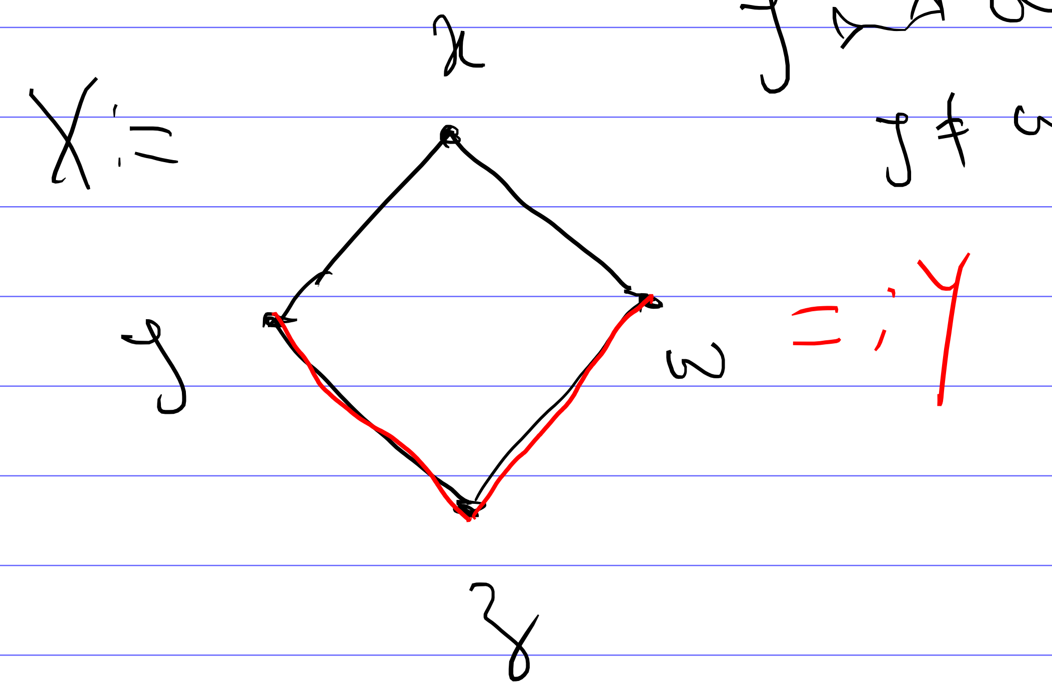
$d(x, y) = d(x, m) + d(m, y)$

$\Rightarrow m \equiv y$.

$m = \mu(x, y, z) \Rightarrow m \in [x, z] \Rightarrow y \in [x, z]$.

$\square$

~~non-eg.~~: "projecting" onto non-convex subgraphs



y and w both minimize the distance to x. $y \neq w$

is not convex

Property:

X median graph, $Y \subseteq X$ convex subgraph, $x \in X$, $J \in \mathcal{H}(X)$. Then:

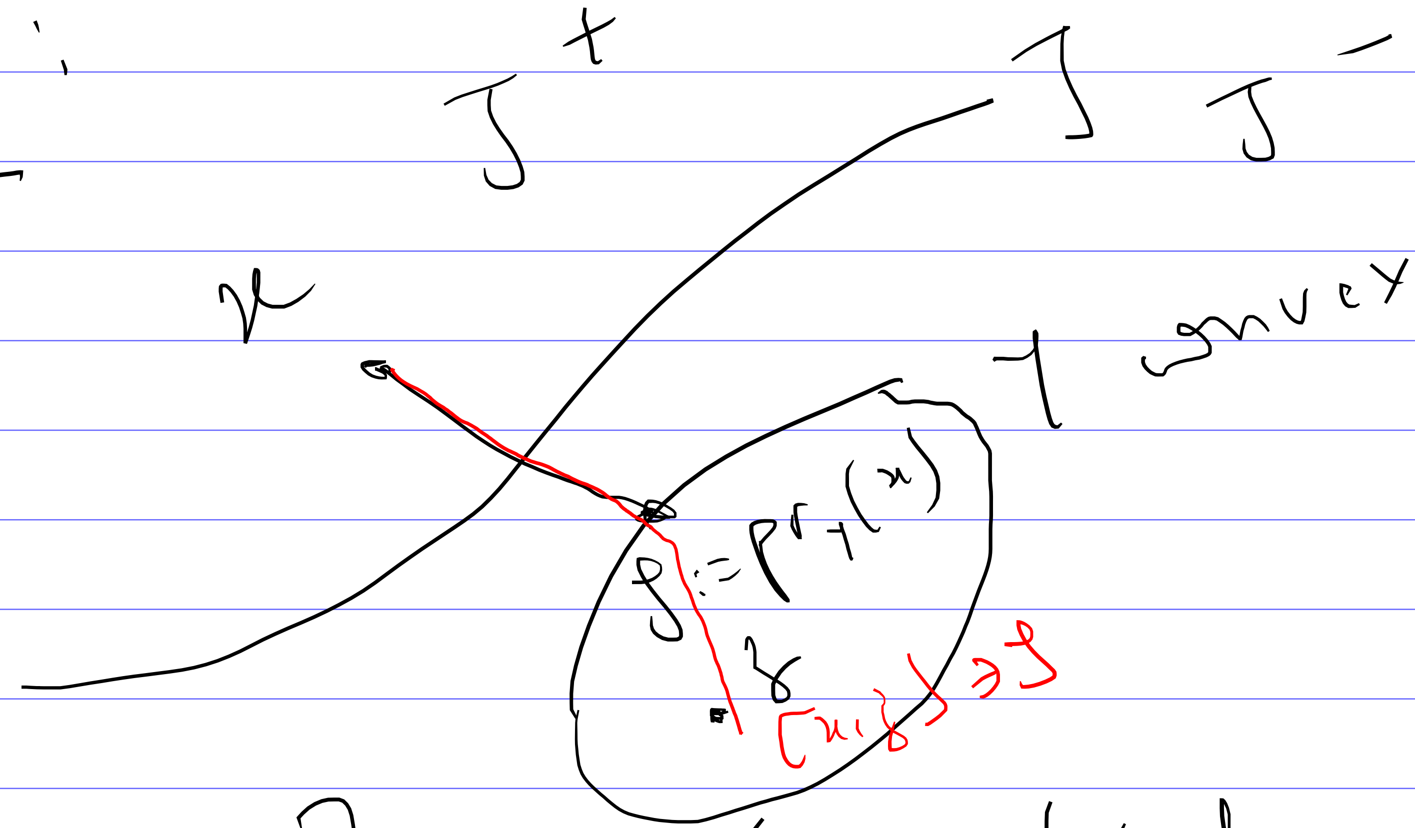
J separates x and $pr_Y(x)$

$\Longleftrightarrow$

J separates x and Y.

Pf.: the trivial case: ($\Leftarrow$)

($\Rightarrow$):



$\Rightarrow \exists \gamma \in [x,y]$ a geodesic that passes through z.

Since $x \in J^+$ and $y \in J^-$, J crosses γ,

Since γ geodesic J crosses γ exactly once,

So $z \in J^-$, $\forall z \in Y$

$\Rightarrow Y \subseteq J^- \Rightarrow J$ separates x and Y. $\square$

Property:

Let X be a median graph.

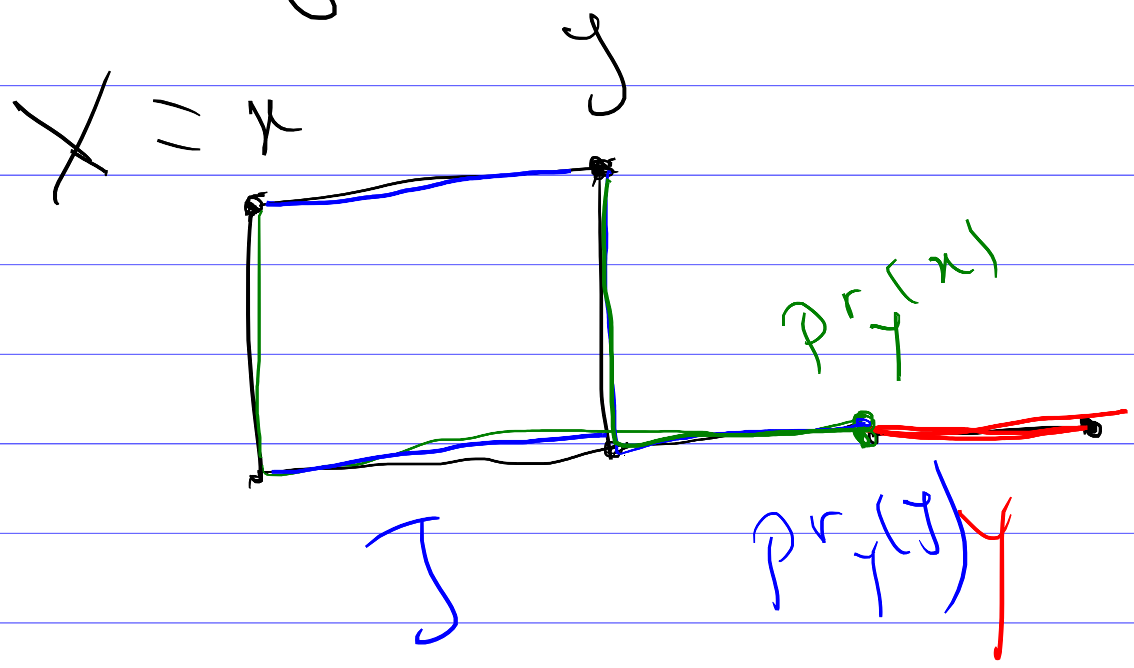
$Y \subseteq X$ convex subgraph

$x, y \in X$.

Then a hyperplane seperates $pr_Y(x)$ and $pr_Y(y)$ iff it seperates x and y and crosses Y.

Moreover, $d(pr_Y(x), pr_Y(y)) \leq d(x,y)$.

counter e.g.:



$x \in J^+$, $y \in J^-$,

but

$pr_Y(x), pr_Y(y) \in J^-$

Problem: J doesn't cross Y.